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Removing and Extracting Features in Images using Mathematical Morphology

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A feature is a local configuration of grey-levels in an image. Removing some types of features can be considered as similar to extracting other types of features, namely those that are not removed. Both operations can be modeled abstractly by an idempotent operator. We examine various meanings of these operations, and link their semantics to algebraic properties of feature operators. Our exposition is informal as far as possible.

Keywords: feature, idempotent operator, morphological filter, opening, open-condensation, open-overcondensation, top-hat.

1. Introduction

A feature means generally a visible local event in an image, and it manifests itself as a peculiar configuration of grey-levels or colours. In mathematical morphology, an image is analysed through its interactions with templates called *structuring elements*. Thus here a feature will represent a template in the image, whose shape (both in space and grey-levels) is linked in a precise way to that of the structuring element.

One can consider that the image is made up of a combination of features and of "non-feature elements", which represent non-local characteristics of the image (such as a constant base grey-level). Extracting one or several types of features means keeping these features and removing other types of features as well as non-feature elements. Removing some types of features means keeping other types of features as well as non-feature elements. Thus feature extraction and feature removal can be considered as equivalent operations in so far as we restrict the image to features, the set of extracted features being the complement of the set of removed features. The main difference between the two lies in the behaviour w.r.t. non-feature elements, namely removing them in feature extraction, and keeping them in feature removal.

From the point of view of mathematical morphology, a fundamental requirement for a "perfect" feature extractor or remover is, as in the case of a "perfect" filter, *idempotence*, which means that applying the operation a second time does not change anymore the image. Indeed, when a feature is removed, this is done completely, so nothing remains to be removed afterwards; on the other hand, an extracted feature is complete, and can thus be extracted from itself. This requirement may exclude certain features which can exist only within a certain context which is not preserved by the feature extraction; for example let us say that a circle is a feature if it is adjacent to a square; extracting the feature could give the circle without the square, and the circle would thus not be preserved by a further feature extraction.

Now a feature remover or extractor can be specified by other properties than simply idempotence. We can for example give additional mathematical properties of this operator, generally of an algebraic nature; in mathematical morphology it is customary to examine properties related to the ordering relation between images and to the composition of operators. We can also describe the types of features to be extracted or removed, and the criteria used for this purpose; this is what we call the *semantics* of feature extraction or removal.

We will describe here several types of operators for feature extraction or removal previously described in the literature, and explain how their semantics relates to their algebraic properties. For the simplicity of exposition, we restrict ourselves here to *anti-extensive* operators, that is those which diminish the object; in other words only positive (bright) features will be extracted or removed. We will consider as feature removers: openings for removing small or isolated features, open-condensations for choosing the "best" opening approximating an object. As feature extractors, we will examine: open-overcondensations which extract features according to both foreground and background templates, and top-hats arising from subtracting the result of an opening from an image.

As far as possible, we give to our exposition a relatively informal style, because the results presented here are already known. The only exception is in Section 3.2, where we present new material. Formal mathematical expositions on morphology can be found in the references. As general expository texts we recommend [4, 12].

2. Feature removal

Here we consider two types of operators: *openings* [11, 12, 4], and *open-condensations* [6]. We first give their formal definition, then describe their possible semantics.

We write $\mathcal{S}$ for the space of pictorial objects (binary figures, grey-level or colour images, etc.). This space is supposed to be ordered by a partial order relation $\leq$ whose meaning is the following: for sets, $X \leq Y$ means that X is included in Y; for grey-level images $X \leq Y$ means that at every point p, the grey-levels $X(p)$ and $Y(p)$ of X and Y at p must satisfy $X(p) \leq Y(p)$; for RGB colour images, at a point p the images X and Y have RGB values $X(p) = \big(X_r(p), X_g(p), X_b(p)\big)$ and $Y(p) = \big(Y_r(p), Y_g(p), Y_b(p)\big)$, and $X \leq Y$ means that at every point p we have $X_r(p) \leq Y_r(p)$, $X_g(p) \leq Y_g(p)$, and $X_b(p) \leq Y_b(p)$.

We assume further that with this ordering $\leq$, $\mathcal{S}$ has the structure of a *complete lattice*. This means that for every family X_i ($i \in I$) of elements of $\mathcal{S}$, there is in $\mathcal{S}$ a *least upper bound* or *supremum* of it, written $\bigvee_{i\in I} X_i$ or $\bigvee\{X_i \mid i \in I\}$, as well as a *greatest lower bound* or *infimum* of it, written $\bigwedge_{i\in I} X_i$ or $\bigwedge\{X_i \mid i \in I\}$. For sets, these operations correspond to the union and intersection; for grey-level images, the supremum and infimum is obtained by taking at each point p the numerical supremum $\sup_{i\in I} X_i(p)$ and infimum $\inf_{i\in I} X_i(p)$ of the respective grey-levels $X_i(p)$ of all images X_i; for RGB images, we take such a supremum and infimum in each R, G, B band at every point p.

For more details and other examples concerning the ordering and complete lattice structure of pictorial objects, the reader is referred to [4, 12].

We give here the formal definition of openings (from [11, 12, 4]) and open-condensations (from [6]); here X, Y, and Z designate arbitrary elements of $\mathcal{S}$:

DEFINITION 1. *An* opening *on $\mathcal{S}$ is an operator $\gamma : \mathcal{S} \to \mathcal{S}$ satisfying the following three requirements:*

1. *γ is* anti-extensive*: $\gamma(X) \leq X$.*
2. *γ is* idempotent*: $\gamma(\gamma(X)) = \gamma(X)$.*
3. *γ is* increasing*: $X \leq Y \implies \gamma(X) \leq \gamma(Y)$.*

DEFINITION 2. *An* open-condensation *on $\mathcal{S}$ is an operator $\gamma : \mathcal{S} \to \mathcal{S}$ satisfying the following three requirements:*

1. *γ is* anti-extensive.
2. *γ is* idempotent.
4. *γ is* condensing*: if $X \leq Y \leq Z$ and $\gamma(X) = \gamma(Z)$, then $\gamma(Y) = \gamma(X)$.*

It should be noted that every opening is an open-condensation. There is an alternate definition of open-condensations (see Proposition 2.2 of [6]):

LEMMA 3. *An operator $\gamma : \mathcal{S} \to \mathcal{S}$ is an open-condensation if and only it satisfies the following two requirements:*

1. *γ is* anti-extensive.

5. $\gamma(X) \leq Y \leq X \implies \gamma(Y) = \gamma(X)$.

Having given the mathematical characteristics of the operators we are interested in, let us explain their possible meanings in terms of features.

2.1. Removing small parts

Historically openings have been associated with the concept of *size distribution* [11]: given a particular size, we apply to our data a kind of sieve that keeps all particles (or parts) bigger or equal to that size, and lets fall all those smaller than that size. Formally, the sieve is an operator γ: if X is the material put into the sieve, $\gamma(X)$ is the material that remains in the sieve. Then this sieve operator γ is *anti-extensive* because it loses matter, and does not add any. It is also *increasing*, because if you put more particles or bigger ones in the sieve, more particles will remain in it.

Composing sieve operations means applying a first sieve, then throwing the matter kept by it into the second one. When the two sieves are the same, the second sieving operation is useless, everything remains the same. Thus the sieve is *idempotent*. If the two sieves correspond to two distinct sizes, then the double sieving is equivalent to a single sieving for the bigger size. Formally, if the two openings γ_s and $\gamma_{s'}$ correspond to the two sizes s and s' respectively, then we have

$$\gamma_s \gamma_{s'} = \gamma_{s'} \gamma_s = \gamma_{\max(s,s')}. \tag{1}$$

A well-know opening is that by a structuring element. Given a structuring element B (a priori B can be any member of our space $\mathcal{S}$ of pictorial objects, but we generally take B to be small), it transforms a picture X into the the supremum of all translates of B that lie inside X. In a formal setting, we write:

$$X \circ B = \bigvee \{\tau(B) \mid \tau \in \mathbf{T},\ \tau(B) \leq X\}. \tag{2}$$

Here $\mathbf{T}$ is the group of all "translations" of our space $\mathcal{S}$. Other openings include taking the supremum of openings by several structuring elements. See [9] for more details. We illustrate in Figure 1 the opening by a structuring element in the case of sets.

The operation where we discard what remains in the sieve and collect what falls from it will be considered in the next section, under the name *top-hat*. The composition of two such operations amounts to putting two sieves on top of each other, and collecting what falls from this combination; this amounts to collecting what falls from the sieve for the smaller size only, so we will have the contrary of (1), namely $\min(s, s')$ instead of $\max(s, s')$.

2.2. Removing isolated particles

This operation was first proposed by Serra in [12], and it was studied in depth in [9]. Here the idea is to remove parts of a picture not because they are too small, but because they are isolated.

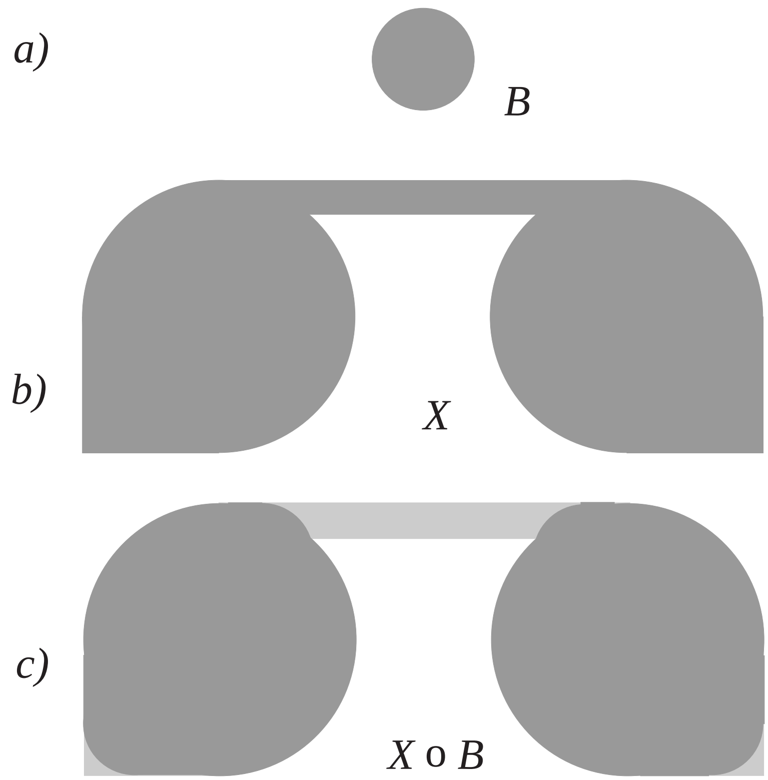

FIGURE 1. Here both images and structuring elements are subsets of the Euclidean plane. a) The structuring element B. b) The figure X. c) The opening $X \circ B$ of X by B (in dark grey) superimposed on X, shown in light grey.

We suppose that an image is made by superposing coloured points; for sets, these are just ordinary points, while for grey-level images they are points with a grey-level attached to it, and for RGB colour images, they are points to which a triple of RGB values is attached. When one superposes two or more coloured points at the same location, their colour or grey-level values are combined by taking the largest grey-level, or the largest value in each RGB band.

We assume a neighbourhood relation $\sim$ on the coloured points constituting the image; this relation is symmetrical in the sense that $p \sim q \iff q \sim p$. Then we transform the image by removing from it every coloured point p such that there is in the image no coloured point q such that $p \sim q$. The remaining image will be constituted of the superposition of all pairs of coloured points p, q from the original image, such that $p \sim q$. This operation is *anti-extensive* because it removes points, and does not add any. It is also *increasing*, because the bigger the original image, the more points it contains, and so the more it can contain pairs p, q with $p \sim q$. Finally, it is idempotent, because the pairs

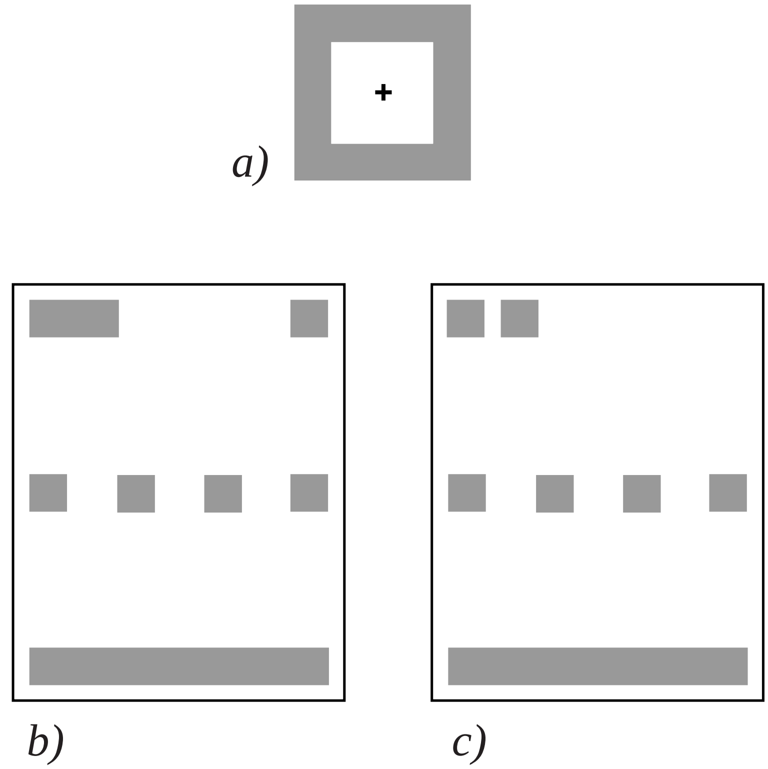

FIGURE 2. Here both images and structuring elements are subsets of the Euclidean plane. a) The ring-shaped structuring element B centered about the origin (marked by a cross). b) The figure X, having 7 connected components. c) The annular opening $X \cap (X \oplus B)$ of X by B; the top left connected component is split into two, and the top right connected component has vanished.

of points p, q with $p \sim q$, which constitute the filtered image, will be preserved under further filtering.

Originally, this operation was devised for sets, and the neighbourhood relation $\sim$ was defined through a structuring element B: $p \sim q$ iff $q - p \in B$; the symmetry of $\sim$ requires B to be symmetric w.r.t. origin. Then the operator transforms a set X into $X \cap (X \oplus B)$, where $X \oplus B$ designates the Minkowski addition of X and B. Experiments were made choosing for B a ring shape, so this new filter was then called an *annular opening*. We illustrate it in Figure 2.

2.3. Adaptive feature removal

Suppose that we demand to remove unwanted features in an image. We can set a size criterion on unwanted features, taking the form "any feature smaller than some unwanted feature must also be unwanted, and should thus be removed".

Assuming as before that we remove positive features, this means that the feature removing operator γ is anti-extensive: $\gamma(X) \leq X$. Here $\gamma(X)$ results from removing unwanted features from X; now an image Y having the same wanted features and non-feature elements as X, but smaller unwanted features, will satisfy:

- $\gamma(X) \leq Y$, because Y contains all wanted features and non-feature elements of X;
- the difference $Y - \gamma(X)$ representing unwanted features of X present in Y, is smaller than $X - \gamma(X)$, which represents all unwanted features of X.

Thus we have $\gamma(X) \leq Y \leq X$. Now we require that those features in Y smaller than unwanted features in X must be removed, that is $\gamma(Y) \leq \gamma(X)$, but wanted features and non-feature elements must be preserved, that is $\gamma(Y) \geq \gamma(X)$. Hence $\gamma(Y) = \gamma(X)$. We have thus obtained the two criteria for an open-condensation given in Lemma 3. Therefore the above size criterion on unwanted features can be interpreted as: *the feature removing operator is an open-condensation*.

Every opening is an open-condensation, and satisfies thus the above criterion. However openings are *increasing*, and this means that a wanted feature in an image remains wanted if we add other features to the image. Now we can envisage the situation where the status of a feature as wanted or unwanted depends on the nature of other features present in the image.

As an example, we consider what we called in [6] a *toggle* of openings: assuming a family of n openings $\gamma_1, \ldots, \gamma_n$, we define the operator γ which transforms X into the "best" among all candidates $\gamma_1(X), \ldots, \gamma_n(X)$. Thus $\gamma(X) = \gamma_i(X)$ for some $i \in \{1, \ldots, n\}$ depending on X. In [6] we gave formal interpretations of the notion of "choosing the best", and obtained precise criteria for γ to be an open-condensation. We give here a simple illustration of this notion. Suppose that we have a valuation operator $\psi : \mathcal{S} \to \mathcal{V}$, where $\mathcal{S}$ is the space of pictorial objects, and $\mathcal{V}$ is a set of "values", totally ordered by an ordering relation written $\preceq$. Thus for every image X, we will compare the "values" $\psi(\gamma_i(X))$ of $\gamma_i(X)$ for $i = 0, \ldots, n$. We choose then $\gamma(X)$ to be the $\gamma_i(X)$ $(i = 0, \ldots, n)$ such that $\psi(\gamma_i(X))$ is maximal for the ordering by $\preceq$:

$$\gamma(X) = \gamma_i(X) \qquad \text{where} \qquad \psi(\gamma_i(X)) = \max\{\psi(\gamma_j(X)) \mid j = 1, \ldots, n\}.$$

When this maximum is attained by two or more candidates $\gamma_{i_1}(X), \ldots, \gamma_{i_k}(X)$, in other words

$$\psi(\gamma_{i_1}(X)) = \cdots = \psi(\gamma_{i_k}(X)) = \max\{\psi(\gamma_j(X)) \mid j = 1, \ldots, n\},$$

we choose between them according to an order of precedence between the indices $i_1, \ldots, i_k$; we can for example assume that the order of precedence between the indices decreases from 1 to n, so we choose the least one among $i_1, \ldots, i_k$:

$$i = \min\big\{u \in \{1, \ldots, n\} \mid \psi(\gamma_u(X)) = \max\{\psi(\gamma_j(X)) \mid j = 1, \ldots, n\}\big\}$$

Now we want $\gamma(X)$ to contain a sizeable part of X, so we require from the valuation ψ the criterion "the bigger, the better", which can be interpreted mathematically as:

$$Y \leq Z \implies \psi(Y) \preceq \psi(Z). \tag{3}$$

Let us show that this guarantees that γ will be an open-condensation. We know that γ is anti-extensive (because each γ_i is), so we have only to satisfy condition 5 of Lemma 3. Let $\gamma(X) = \gamma_i(X)$; thus for $j \neq i$, we have either:

- $\psi(\gamma_j(X)) \prec \psi(\gamma_i(X))$, or
- $\psi(\gamma_j(X)) = \psi(\gamma_i(X))$ and $i < j$.

Now let Y satisfy $\gamma(X) \leq Y \leq X$; as $\gamma(X) = \gamma_i(X)$ and γ_i is an opening, we get $\gamma_i(Y) = \gamma_i(X)$; for $j \neq i$, $Y \leq X$ and the fact that γ_j is increasing imply that $\gamma_j(Y) \leq \gamma_j(X)$. Thus from (3) we get:

$$\psi(\gamma_i(Y)) = \psi(\gamma_i(X)) \qquad \text{and} \qquad \psi(\gamma_j(Y)) \preceq \psi(\gamma_j(X)) \quad (j \neq i).$$

Hence we have either:

- $\psi(\gamma_j(Y)) \prec \psi(\gamma_i(Y))$, or
- $\psi(\gamma_j(Y)) = \psi(\gamma_i(Y))$ and $i < j$.

Therefore we will get $\gamma(Y) = \gamma_i(Y) = \gamma_i(X) = \gamma(X)$ and γ is an open-condensation.

A simple example of the criterion "the bigger, the better" (3) is given by choosing the $\gamma_i(X)$ having greatest size (for discrete figures), or greatest area/volume (for 2D/3D Euclidean figures). We illustrate such a "toggle" of openings in Figure 3

3. Feature extraction

Openings and open-condensations could be considered together, since the latter is just a generalization of the former; in some sense an open-condensation can be considered as an "adaptive" opening.

We will now describe two types of feature extractors that are very different in their conception: one of them is a generalization of the opening by a structuring element that uses a second structuring element as the "negative" part of the feature, and the other will take the arithmetic difference between the original image and the result of a feature remover, such as an opening. Their mathematical formalizations have nothing in common, so we will consider them separately.

3.1. Features with positive and negative aspects

The basic idea here is that a shape is characterized by specifying not only points that must belong to it, but also points that may not belong to it. A

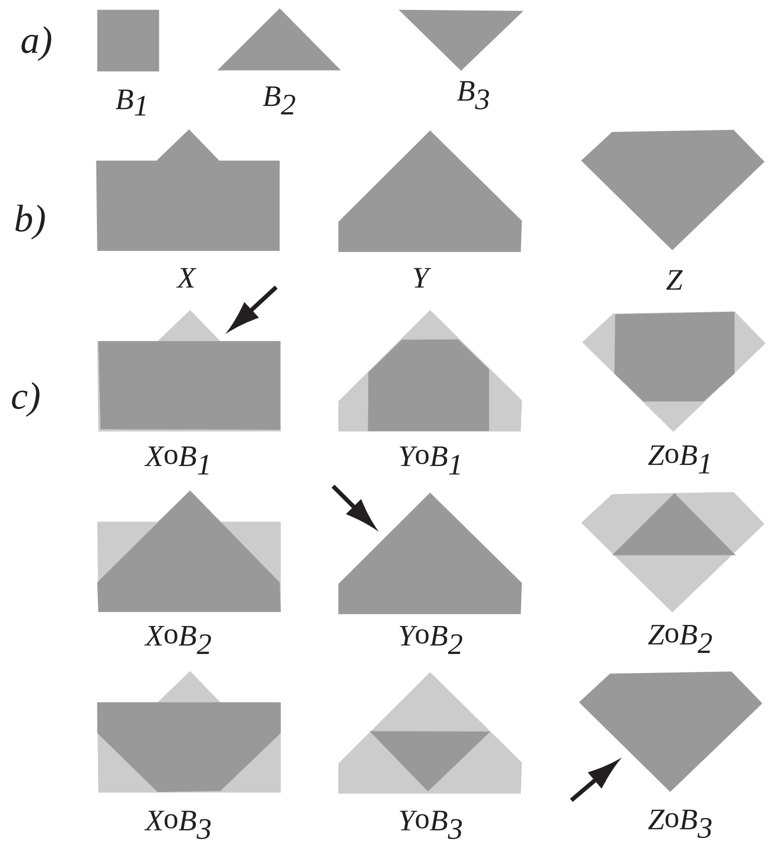

FIGURE 3. Here both images and structuring elements are subsets of the Euclidean plane. a) Three structuring elements B_1, B_2, B_3. b) Three figures X, Y, Z. c) The openings of each of X, Y, Z by B_1, B_2, B_3 respectively (in dark grey, superimposed on the original figure X, Y, Z, shown in light grey); for each figure, we indicate by an arrow the opening having the greatest area. The latter is selected as the result of the toggle γ; thus $\gamma(X) = X \circ B_1$, while $\gamma(Y) = Y \circ B_2$, and $\gamma(Z) = Z \circ B_3$.

well-known example is considered in [1]. We consider as objects all subsets of the Euclidean plane; we call them *figures*. For any figure F and point p in that plane, we write F_p for the translate of F by p. Let A be a square shape and let X be a figure. We want to find all positions where there is in X a square which is a translate of A. The Minkowski subtraction $X \ominus A$, which consists of all points p such that $A_p \subseteq X$, will indeed give all positions p where a translate of A is included in X, but at such a position X does not necessarily take the shape of A; for example a rectangle bigger than A will give several such points p: see Figure 4. We need in fact to have at position p the square A_p included

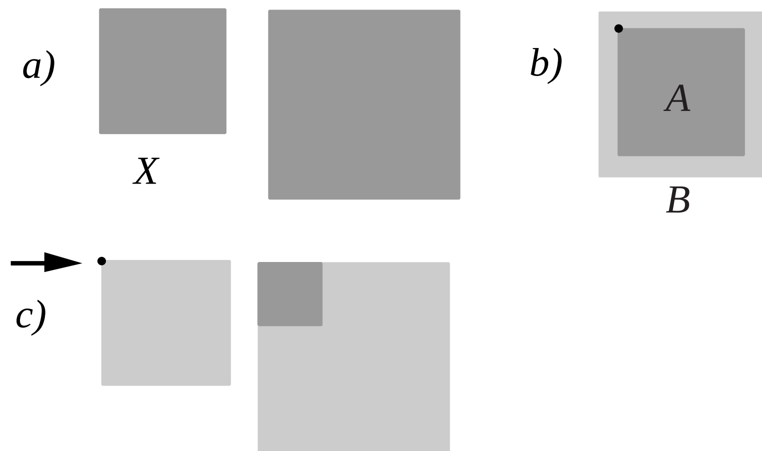

FIGURE 4. Here both images and structuring elements are subsets of the Euclidean plane. a) Figure X. b) The two structuring elements A and B (dark and light grey, resp.); the origin, located at the top left corner of A, is marked by a dot. c) Superimposed on X (displayed in grey), we show (in dark grey) $X \ominus A$, giving all locations where a translate of A is included in X. The hit-or-miss transform $(X \ominus A) \cap (X^c \ominus B)$ of X by (A, B) gives all positions p where the figure X contains a square A_p surrounded by a strip B_p in the background X^c; here it reduces to a single point, indicated with an arrow.

in figure X, but at the same time A_p must be surrounded by points which do not belong to X. Let B be a narrow strip surrounding A (we have $A \cap B = \emptyset$); write X^c for the complement of X (it is called the background). Then we will have a square shape in X at every position p where $A_p \subseteq X$ and $B_p \subseteq X^c$. We get the *hit-or-miss transform of X by* (A, B):

$$(X \ominus A) \cap (X^c \ominus B) = \{p \mid A_p \subseteq X \text{ and } B_p \subseteq X^c\}. \tag{4}$$

We see in Figure 4 that this gives indeed the precise locations where X takes the shape of a square which is a translate of A.

This example shows that the notion of a square shape is characterized both by a positive aspect (the points of the square) and a negative aspect (the surrounding points which are not in the square). In order to display the squares located by (4), we take the Minkowski addition of this set and A:

$$\big[(X \ominus A) \cap (X^c \ominus B)\big] \oplus A = \bigcup \{A_p \mid A_p \subseteq X \text{ and } B_p \subseteq X^c\}. \tag{5}$$

Such an operation is not restricted to our particular example where A is the square and B is the surrounding strip; it can be generally defined for any two disjoint structuring elements A and B. Now take $C = B^c$; note that since $A \cap B = \emptyset$, we must have $A \subseteq C$. The condition $B_p \subseteq X^c$ can be rewritten

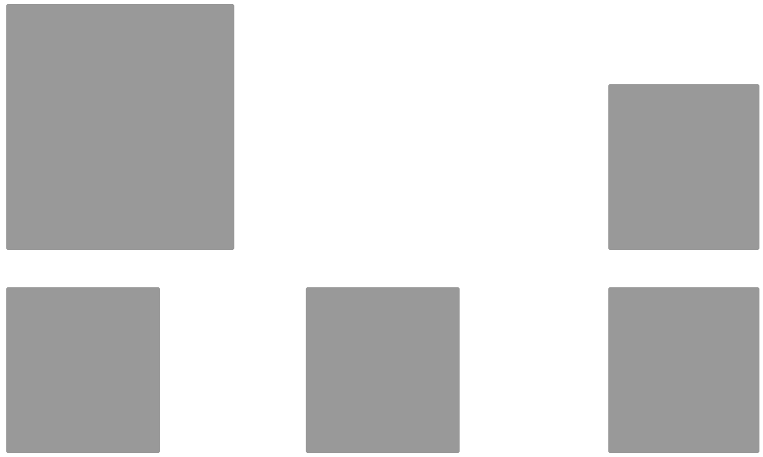

FIGURE 5. Write γ for the operation of (5) with the structuring elements of Figure 4. Two figures X, Y such that $\gamma(X) \subseteq Y \subseteq X$ will satisfy $\gamma(X) \subseteq \gamma(Y)$.

$X \subseteq C_p$, and the above equation becomes:

$$X \oslash (A, C) = \big[(X \ominus A) \cap (X^c \ominus C^c)\big] \oplus A = \bigcup\{A_p \mid A_p \subseteq X \subseteq C_p\}. \qquad (6)$$

This new operation $\oslash$ was defined in [7]. Note that for $B = \emptyset$ (that is C is the whole space), (5)–(6) reduce to the opening $X \circ A$ of X by A. We have thus a generalization of the opening by a structuring element.

Such an operation extracting features defined in terms of a positive shape A and a negative shape B is certainly anti-extensive, but it has also another property: consider a figure X transformed by this feature extractor into Z, which is a union of translates of A inside X to which correspond translates of B outside X; for any figure Y such that $Z \subseteq Y \subseteq X$, Y will contain these translates of A (because $Z \subseteq Y$), and the corresponding translates of B, being outside X, will be outside Y (because $Y \subseteq X$). Thus the features present in X are also present in Y, and the result of the feature extractor on Y will contain Z; it can even be larger than Z, as shown in Figure 5. If we designate by γ this feature extractor, we can write

$$\gamma(X) \subseteq Y \subseteq X \implies \gamma(X) \subseteq \gamma(Y).$$

We give now a general definition (from [7]) of the type of mathematical operation involved. We do not restrict ourselves to sets, but our pictorial objects belong to the ordered set $\mathcal{S}$, as explained in the previous section:

DEFINITION 4. *An* open-overcondensation *on $\mathcal{S}$ is an operator $\gamma : \mathcal{S} \to \mathcal{S}$ satisfying the following three requirements:*

1. γ *is* anti-extensive.

2. γ *is* idempotent.

6. γ *is* overcondensing*: if* $X \leq Y \leq Z$ *and* $\gamma(X) = \gamma(Z)$*, then* $\gamma(X) \leq \gamma(Y)$.

It should be noted that every open-condensation, in particular every opening, is an open-overcondensation. There is an alternate definition of open-overcondensations (see Lemma 2.3 of [7]):

LEMMA 5. *An operator* $\gamma : \mathcal{S} \to \mathcal{S}$ *is an open-overcondensation if and only it satisfies the following two requirements:*

1. γ *is anti-extensive.*

7. $\gamma(X) \leq Y \leq X \implies \gamma(X) \leq \gamma(Y)$.

A typical open-overcondensation is the generalization to an ordered set $\mathcal{S}$ of (6), namely

$$X \oslash (A, C) = \bigvee \{\tau(A) \mid \tau \in \mathbf{T},\ \tau(A) \leq X \leq \tau(C)\} \qquad (A \leq C). \qquad (7)$$

Here (as in (2)) $\mathbf{T}$ is the group of all "translations" of our space $\mathcal{S}$. Note that when C is the greatest element 1 of $\mathcal{S}$ (namely for sets, 1 is the whole space, while for grey-level/colour images, 1 is the image having the greatest grey-level/colour on all points), $X \oslash (A, C)$ reduces to $X \circ A$. Other open-overcondensations include taking the supremum of such operations (7) with several pairs of structuring elements (A, C) satisfying $A \leq C$; this means in practice extracting several types of features from the image. These two facts imply that open-overcondensations represent a generalization of openings, where we add to each structuring element a second one representing the negative aspect of the shape. See [7] for more details.

3.2. Extracting features by subtracting an opening

Given an anti-extensive feature remover γ, we can in some way extract the features removed by γ from an image X by taking the arithmetic difference $X - \gamma(X)$. For grey-level images on a space E, this means that we take the image having at every point $p \in E$ the difference $X(p) - \gamma(X)(p)$ of grey-levels of X and $\gamma(X)$ at p; for colour images, we take the arithmetic difference between the R, G, and B levels of X and $\gamma(X)$ at p. For subsets of space E, this difference must be interpreted as the set subtraction $X \setminus \gamma(X)$; indeed every set $A \subseteq E$ can be identified with its characteristic function $\chi_A : E \to \{0, 1\}$, and for $\gamma(X) \subseteq X$ we have $\chi_{X \setminus \gamma(X)} = \chi_X - \chi_{\gamma(X)}$.

There is a general technical problem of specifying when two images can be arithmetically added or subtracted. For example binary images with values in $\{0, 1\}$ associated to points (in other words characteristic functions of sets) cannot always be added or subtracted, because this could lead to values 2 or

-1, outside the range $\{0,1\}$. For grey-level or colour images, mathematical morphology requires the set of grey-levels or RGB values to be closed, so one generally chooses for it the extended real line $\overline{\mathbf{R}} = \mathbf{R} \cup \{\pm\infty\}$, the extended integer line $\overline{\mathbf{Z}} = \mathbf{Z} \cup \{\pm\infty\}$, a closed real interval $[a, b]$, or an integer interval $[a \ldots b] = [a, b] \cap \mathbf{Z}$. For the extended real or integer line, the problem is dealing with the subtraction $+\infty - \infty$, while for a real or integer interval, we can by adding or subtracting images obtain values outside the interval. We will exclude here infinite values, and assume a general principle of the form: *addition and subtraction of images is always possible, as long as it remains within bounds.* More formally, the object space $\mathcal{S}$ (i.e., the family of all images) satisfies the following: given

$$A_0, A, A_1, B_0, B, B_1 \in \mathcal{S} \quad \text{such that} \quad A_0 \leq A \leq A_1 \text{ and } B_0 \leq B \leq B_1,$$

we have

$$A_0 + B_0, A_1 + B_1 \in \mathcal{S} \implies A + B \in \mathcal{S} \text{ and } A_0 + B_0 \leq A + B \leq A_1 + B_1,$$

and

$$A_0 - B_1, A_1 - B_0 \in \mathcal{S} \implies A - B \in \mathcal{S} \text{ and } A_0 - B_1 \leq A - B \leq A_1 - B_0.$$

This principle is for example satisfied when $\mathcal{S}$ consists of all numerical functions with values in an interval of $\mathbf{R}$ or $\mathbf{Z}$ (grey-level images with bounded real or integer grey-levels), or vector-valued functions with values in an interval of $\mathbf{R}^m$ or $\mathbf{Z}^m$ (e.g., for $m = 3$, RGB colour images with bounded real or integer RGB values).

Note that for grey-level and colour images, $X - \gamma(X)$ is an image having non-negative grey-levels (resp. RGB values) at each point. In a more formal way, we write 0 for the image which is the neutral for addition ($X + 0 = 0 + X = X$ for every image X): it has zero grey-level (resp. RGB value) at each point; then $\gamma(X) \leq X$ leads to $X - \gamma(X) \geq 0$. Since we will examine the possibility of idempotence of the operator $X \to X - \gamma(X)$, it is possible (but not necessary) to restrict the scope of γ (in other words the object space $\mathcal{S}$) to *positive images*, that is images $X \geq 0$. In other words, we can assume that the set of grey-levels or RGB values consists only of non-negative numbers.

Let us write **id** for the identity operator $X \to X$ on $\mathcal{S}$; then we will write $\mathbf{id} - \gamma$ for the operator $X \to X - \gamma(X)$. Such an operator has sometimes been called a *top-hat* in the literature. Indeed, as it extracts a particular type of feature and removes other features as well as non-feature elements, it tends to show isolated features standing on a zero background, and the grey-level (or colour) profile of such a feature looks then like a top-hat. In some sense, every feature extractor, in particular the open-overcondensation defined in (7), could be called a "top-hat".

If we return to the sieve analogy used above for openings, the sieve performs on the matter put into it two complementary operations: first an opening γ that keeps in the sieve all particles larger than the size corresponding to the sieve, second a top-hat $\mathbf{id} - \gamma$ that collects all smaller particles falling from the sieve. The composition of two top-hats amounts to putting the two sieves on top of each other, and collecting what falls from this combination. When the two sieves are the same, the second sieve is useless, everything falls through it; thus a top-hat should be idempotent. If the two sieves correspond to two distinct sizes, then this double sieving amounts to collecting what falls from the sieve for the smaller size only, so we will have the contrary of (1):

$$(\mathbf{id} - \gamma_s)(\mathbf{id} - \gamma_{s'}) = (\mathbf{id} - \gamma_{s'})(\mathbf{id} - \gamma_s) = \mathbf{id} - \gamma_{\min(s,s')}. \qquad (8)$$

When the two openings γ_s and $\gamma_{s'}$ satisfy (1), we do not necessarily get (8); in particular, for an opening γ, $\mathbf{id} - \gamma$ is generally not idempotent. We will give below conditions for this idempotence.

Write $\mathbf{0}$ for the constant operator $X \mapsto 0$ on $\mathcal{S}$. We have a general result concerning conditions for satisfying (8):

LEMMA 6. *Let γ, γ' be anti-extensive operators $\mathcal{S} \to \mathcal{S}$. Then:*

(i) $(\mathbf{id} - \gamma)(\mathbf{id} - \gamma') = \mathbf{id} - \gamma'$ *iff* $\gamma(\mathbf{id} - \gamma') = \mathbf{0}$.

(ii) $(\mathbf{id} - \gamma)(\mathbf{id} - \gamma') = \mathbf{id} - \gamma$ *iff* $\gamma(\mathbf{id} - \gamma') = \gamma - \gamma'$.

PROOF We have

$$(\mathbf{id} - \gamma)(\mathbf{id} - \gamma') = \mathbf{id}(\mathbf{id} - \gamma') - \gamma(\mathbf{id} - \gamma') = \mathbf{id} - \gamma' - \gamma(\mathbf{id} - \gamma').$$

Thus $(\mathbf{id} - \gamma)(\mathbf{id} - \gamma') = \mathbf{id} - \gamma'$ iff $\mathbf{id} - \gamma' - \gamma(\mathbf{id} - \gamma') = \mathbf{id} - \gamma'$, that is $\gamma(\mathbf{id} - \gamma') = \mathbf{0}$, and *(i)* holds. On the other hand, $(\mathbf{id} - \gamma)(\mathbf{id} - \gamma') = \mathbf{id} - \gamma$ iff $\mathbf{id} - \gamma' - \gamma(\mathbf{id} - \gamma') = \mathbf{id} - \gamma$, that is $\gamma(\mathbf{id} - \gamma') = \gamma - \gamma'$, and *(ii)* holds. *Q.E.D.*

A particular case of *(i)* is that $\mathbf{id} - \gamma$ is idempotent iff $\gamma(\mathbf{id} - \gamma) = \mathbf{0}$.

We introduced the sieving analogy for openings. So we could expect that in order to use it for a top-hat $\mathbf{id} - \gamma$, we should assume that γ is an opening. Indeed, we will obtain some results that are valid only with that assumption.

We recall from [9] that an *invariant* of γ is some $X \in \mathcal{S}$ such that $\gamma(X) = X$, and we write $\mathrm{Inv}(\gamma)$ for the family of all invariants of γ. Now every opening γ is uniquely characterized by its family $\mathrm{Inv}(\gamma)$ of invariants (Corollary 2.4 of [9]), and $\mathrm{Inv}(\gamma)$ is closed under the supremum operation $\bigvee$ (Proposition 2.2 of [9]).

Before giving our next condition for the idempotence of a top-hat $\mathbf{id} - \gamma$, we must make an assumption on $\mathcal{S}$ which is analogous to the Archimedes axiom for reals, saying that for two reals $a \geq 0$ and $b > 0$ there is a natural integer n such that $nb > a$. We require that for $A, B \in \mathcal{S}$ such that $A \geq 0$, $B \geq 0$ and $B \neq 0$, there is a natural integer n such that either $nB \notin \mathcal{S}$, or $nB \in \mathcal{S}$ but $nB \not\leq A$ (NB. nB is defined here as the addition $B + \cdots + B$ of n times B). Note that we have added the possibility $nB \notin \mathcal{S}$ in order to take into account the case where $\mathcal{S}$ is a family of numerical functions having values in a bounded range, in which case adding successively B to itself can lead to values outside the bounds; the same would happen with the Archimedes axiom if we had restricted its scope to reals in an interval I: there is some $n \in \mathbf{N}$ such that either $nb \notin I$, or $nb \in I$ but $nb > a$. We say then that $\mathcal{S}$ is *Archimedean.* This axiom is always satisfied when $\mathcal{S}$ is a family of numerical functions with values in a subset of $\mathbf{R}$ (namely, grey-level images), or vector functions with values in a subset of $\mathbf{R}^m$ (for example RGB colour images for $m = 3$, multimodal images, etc.); in particular it is satisfied with $\mathcal{S}$ consisting of sets (which can be interpreted as functions with binary values in $\{0, 1\}$). It excludes only exotic object spaces, such as functions whose values include numbers which are "infinitely larger" than other ones (such as in non-standard arithmetic).

We will now give a sufficient condition for an opening γ to give an idempotent top-hat $\mathbf{id} - \gamma$. We assume that $\mathcal{S}$, besides being Archimedean, consists only of positive images:

THEOREM 7. *Let $\mathcal{S}$ be Archimedean and such that for every $X \in \mathcal{S}$ we have $X \geq 0$. Let γ be an opening on $\mathcal{S}$ such that for every $B \in \mathrm{Inv}(\gamma)$ and every integer $n \geq 1$ with $nB \in \mathcal{S}$ we must have $nB \in \mathrm{Inv}(\gamma)$. Then $\mathbf{id} - \gamma$ is idempotent, and for every opening γ' satisfying $\gamma'\gamma = \gamma'$ we have $(\mathbf{id} - \gamma')(\mathbf{id} - \gamma) = \mathbf{id} - \gamma$.*

PROOF By Lemma 6 *(i)*, we have only to satisfy that $\gamma(\mathbf{id} - \gamma) = \mathbf{0}$, in other words every $X \in \mathcal{S}$ gives $\gamma(X - \gamma(X)) = 0$. Let $Z = \gamma(X - \gamma(X))$; write $Y = X - \gamma(X)$, so $Z = \gamma(Y)$. As γ is idempotent, we have $Z = \gamma(Z)$, that is $Z \in \mathrm{Inv}(\gamma)$. We show now by induction that for every integer $n \geq 0$ we have $nZ \in \mathcal{S}$ and $nZ \leq X$. This is certainly true for $n = 0$: $X \geq 0 = 0Z$. Supposing the property true for n, we show it for $n+1$. If $nZ \in \mathcal{S}$ and $nZ \leq X$, we must have $nZ \in \mathrm{Inv}(\gamma)$: this comes from hypothesis for $n \geq 1$, while for $n = 0$, we have $\gamma(0) \leq 0$ by anti-extensivity of γ and $0 \leq \gamma(0)$ by hypothesis, so $0Z = 0 \in \mathrm{Inv}(\gamma)$. Now as γ is increasing, we get $nZ = \gamma(nZ) \leq \gamma(X)$; as γ is anti-extensive, we have $Z \leq Y = X - \gamma(X)$; since $0 \leq nZ \leq \gamma(X)$ and $0 \leq Z \leq X - \gamma(X)$, we deduce by summing both inequalities that $(n+1)Z \in \mathcal{S}$ and $(n+1)Z \leq X$. By induction hypothesis, the property is thus true for every integer $n \geq 0$; as $Z \geq 0$, the axiom that $\mathcal{S}$ is Archimedean implies that $Z = 0$. Therefore $\gamma(X - \gamma(X)) = 0$ and $\mathbf{id} - \gamma$ is idempotent.

Given an opening γ' satisfying $\gamma'\gamma = \gamma'$, the equality $\gamma(\mathbf{id} - \gamma) = \mathbf{0}$ gives

$$\gamma'(\mathbf{id} - \gamma) = \gamma'\gamma(\mathbf{id} - \gamma) = \gamma'\mathbf{0} = \mathbf{0},$$

and Lemma 6 *(i)* implies then that we have $(\mathbf{id} - \gamma')(\mathbf{id} - \gamma) = \mathbf{id} - \gamma$. *Q.E.D.*

In [6] we called a *para-opening* an increasing and anti-extensive operator γ such that $\mathbf{id} - \gamma$ is idempotent. We showed in particular (see Proposition 4.6 of [6]) that a supremum $\bigvee_{i \in \mathcal{I}} \gamma_i$ of para-openings γ_i is a para-opening.

When $\mathcal{S}$ is the set of parts of a space E of points, Theorem 7 holds for every opening, because the hypothesis is always satisfied. Indeed for $B \subseteq E$, either $B = \emptyset$ and then $nB = B$ for every integer $n \geq 1$, or $B \neq \emptyset$ and then we have $1B = B$ but $nB \notin \mathcal{S}$ for $n \geq 2$. For grey-level images, the result holds for *flat* openings [2, 3], because here the image is processed by operating on "grey-level slices" (for a grey-level t, such a slice consists of all points having grey-level $\geq t$), and so for every $B \in \mathrm{Inv}(\gamma)$ and every scalar λ with $\lambda B \in \mathcal{S}$ we have $\lambda B \in \mathrm{Inv}(\gamma)$. The reader is refered to pp. 119,120 of [6] for a deeper explanation of this point.

Note that the hypothesis of Theorem 7 is not a necessary condition for the idempotence of $\mathbf{id} - \gamma$. We have the following counterexample with grey-level functions: we take the opening γ whose invariants are all non-negative functions $f : \mathbf{R}^2 \to [0, 2]$ such that there is an open subset G of $\mathbf{R}^2$ with which $f(x) = 1$ for an irrational $x \in G$, $f(x) \geq 1$ for a rational $x \in G$, $f(x) = 0$ for an irrational $x \notin G$, and $f(x) \geq 0$ for a rational $x \notin G$; then clearly $X - \gamma(X)$ has value 0 on every rational point, so $\gamma(X - \gamma(X)) = 0$; however the hypothesis of Theorem 7 is not satisfied.

Note that for grey-level functions, if γ is the opening by a structuring function B (transforming X into $X \circ B$, see (2)), where B is a compactly supported function, then it can be shown that $\mathbf{id} - \gamma$ is idempotent if and only if B is constant on its support, in other words the opening is flat, in which case the hypothesis of Theorem 7 is satisfied. This result is illustrated in Figure 6 of [6], where we give an extreme example of an infinitely descending sequence $(\mathbf{id} - \gamma)^n(X)$.

We would like to have conditions ensuring that openings satisfying (1) will also satisfy (8). Given two openings γ, γ', we have (see Proposition 2.3 of [9]):

$$\gamma' \leq \gamma \iff \gamma'\gamma = \gamma' \iff \gamma\gamma' = \gamma' \iff \mathrm{Inv}(\gamma') \subseteq \mathrm{Inv}(\gamma). \tag{9}$$

If $\mathbf{id} - \gamma$ is idempotent, then we see from the proof of Theorem 7 that (9) gives $(\mathbf{id} - \gamma')(\mathbf{id} - \gamma) = \mathbf{id} - \gamma$. However, this does generally not give $(\mathbf{id} - \gamma)(\mathbf{id} - \gamma') = \mathbf{id} - \gamma$. As seen in Lemma 6 *(ii)* the necessary and sufficient condition is $\gamma(\mathbf{id} - \gamma') = \gamma - \gamma'$, which is generally not satisfied, even when $\mathcal{S}$ is the set of parts of a space of points: we give an example in Figure 6.

Note that for sets, it is easily seen that every X gives always

$$\gamma\big(X \setminus \gamma'(X)\big) \subseteq \gamma(X) \cap \big(X \setminus \gamma'(X)\big) = \gamma(X) \setminus \gamma'(X) \subseteq X \setminus \gamma'(X),$$

in other words:

$$\gamma(\mathbf{id} - \gamma') \leq \gamma - \gamma' \leq \mathbf{id} - \gamma'.$$

Applying γ to every term of this inequality, this gives

$$\gamma(\mathbf{id} - \gamma') = \gamma(\gamma - \gamma'). \tag{10}$$

We conjecture that in order to obtain $\gamma(\mathbf{id} - \gamma') = \gamma - \gamma'$ from (9), it must be necessary to make assumptions on γ and γ' related to connectedness, where the latter notion does not need to be taken in a topological sense. Indeed, in mathematical morphology purely algebraic axioms have been given for the

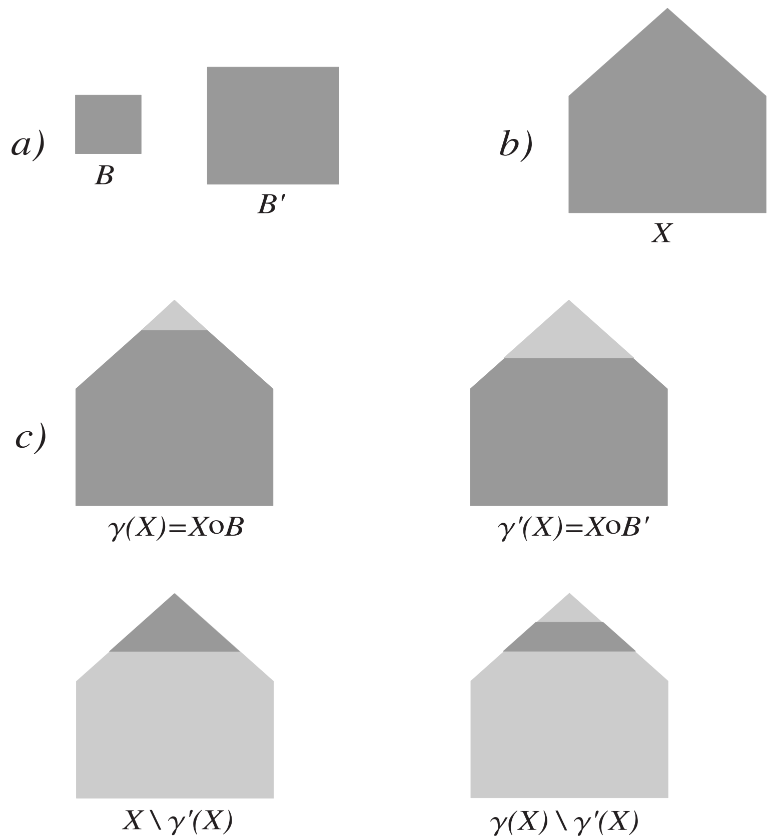

FIGURE 6. Here both images and structuring elements are subsets of the Euclidean plane. a) The two structuring elements B and B'; the openings γ, γ' defined by $\gamma(X) = X \circ B$ and $\gamma'(X) = X \circ B'$ satisfy (9). b) The set X. c) Superimposed on X (light grey), the sets $\gamma(X)$, $\gamma'(X)$, $X \setminus \gamma'(X)$, and $\gamma(X) \setminus \gamma'(X)$ (in dark grey); we have $\gamma\big(X \setminus \gamma'(X)\big) = \emptyset \neq \gamma(X) \setminus \gamma'(X)$.

abstract notion of a connectivity class (see Chapter 2 of [12], Subsection 1.1 of [5], and [8]): in a space E a *connectivity class* is a family $\mathcal{C}$ of subsets of E (the elements of $\mathcal{C}$ are the "connected" subsets of E), such that (a) $\emptyset \in \mathcal{C}$ and $\forall x \in E$, $\{x\} \in \mathcal{C}$; (b) for $\mathcal{B} \subseteq \mathcal{C}$ such that $\bigcap \mathcal{B} \neq \emptyset$, we have $\bigcup \mathcal{B} \in \mathcal{C}$. These axioms include as particular cases connectedness in a topological space and in a graph.

For example suppose that for every set X, $\gamma'(X)$ is a union of connected components of X, and for every invariant B of γ, all connected components of B are invariants of γ; then $\gamma(X) \setminus \gamma'(X) = \gamma(X) \setminus \gamma'(\gamma(X))$ will consist of a union of connected components of $\gamma(X)$ (namely those not selected by γ'), and so it will be invariant under γ, and (10) will give

$$\gamma\big(X \setminus \gamma'(X)\big) = \gamma\big(\gamma(X) \setminus \gamma'(X)\big) = \gamma(X) \setminus \gamma'(X),$$

that is $\gamma(\mathbf{id} - \gamma') = \gamma - \gamma'$. Therefore $(\mathbf{id} - \gamma')(\mathbf{id} - \gamma) = \mathbf{id} - \gamma$ in this case.

It seems that we cannot find a property related to order, like condensation or overcondensation, for the feature extractor $\mathbf{id} - \gamma$. However, such operators can be used in the construction of open-condensations, as shown in Section 4 of [6].

4. Conclusion

We have reviewed the concrete meaning (semantics) and algebraic properties of several types of feature removers and feature extractors. Our collection is certainly far from complete, especially since we restricted ourselves to anti-extensive operators, in other words we remove only positive features or non-feature elements from the image. In fact, this article is only an informal and partial introduction to the theory of morphological feature extraction/removal. There is a wider theory of idempotent operators having various properties related to order; this is a subject of ongoing research, with new works published every year. See [4, 12] for some well-known results.

The theory of openings [4, 9, 11, 12] is classical, although new types of openings can be invented, with a new semantical interpretation (this happened for example with annular openings, see Subsection 2.2). Although more recent, the theory of open-overcondensations [7] is very regular and has many parallels with that of openings. However, the intermediate concept of open-condensation does not lend itself so easily to such a regular theory; it gives rather a family of tools for constructing open-condensations [6]; these tools have a somewhat algorithmic nature [10] adapted to the notion of progressive coding of pictures. Finally the theory of top-hats of the form $\mathbf{id} - \gamma$ for an opening γ has never been systematically studied; this is partly due to the difficulty arising from a combination of a morphological operator with an arithmetic subtraction. In the case of sets, this subtraction becomes a set difference, so it can be more easily interpreted within the framework of mathematical morphology, but even in this case severe restrictions seem necessary in order to obtain the sieving property (8) corresponding to that (1) for openings.

To our knowledge, Theorem 7 has never been stated explicitly in the literature, although some forms of it were already known: in the case of sets, it is trivial to show that $\mathbf{id} - \gamma$ is idempotent for every increasing and anti-extensive γ, while for grey-level functions, the idempotence of $\mathbf{id} - \gamma$ for a flat opening γ is a kind of "folk theorem", and its proof is given on pp. 119,120 of [6].

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